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# SHORTER COMMUNICATIONS

## EFFECTS OF VISCOUS DISSIPATION ON COMBINED FREE AND FORCED CONVECTION THROUGH VERTICAL CONCENTRIC ANNULI

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### NOMENCLATURE

$Eck$, Eckert number ($= U^2/C_p\Delta T$) [dimensionless];
$L$, pressure drop parameter ($= -D_h^2\{dp/dz + \rho_w g\}/4\mu U$) [dimensionless];
$M$, viscous dissipation parameter ($= Eck/Re$) [dimensionless];
$Nu$, Nusselt number ($= hD_h/k$) [dimensionless];
$R$, dimensionless radius ($= 2r/D_h$);
$R_i$, dimensionless inner radius ($= 2r_i/D_h$);
$R_0$, dimensionless outer radius ($= 2r_0/D_h$);
$Ra$, Rayleigh number ($= \rho^2\beta g C_p C D_h/16\mu k$) [dimensionless];
$Re$, Reynolds number ($= UD_h/\nu$) [dimensionless];
$U$, average axial velocity;
$\nu, \beta$, fluid properties;
$\mu, \rho$, fluid properties;
$\eta$, $(Ra)^{\frac{1}{4}}$;
$\lambda$, $r_i/r_0$, radius ratio [dimensionless];
$\phi$, temperature function ($= 4k\{T - T_w\}/\{\rho U C_p C D_h^2\}$) [dimensionless].

### INTRODUCTION

THE OMISSION of viscous dissipation in the thermal energy balance for viscous flow would be unrealistic from the physics of fluids. For external flows, the effect of viscous dissipation is found to be quite significant because of the energy generated in the boundary layer, and the skin temperatures that are attained at very high velocities. Several studies have been made in this regard because the phenomena of 'aerodynamic heating' at high Mach numbers can cause severe problems due to the temperature limitations of structural materials commonly used in the manufacture of aircraft parts and missiles.

The study of the effects of viscous dissipation in internal laminar flows can be divided into three parts, (i) forced convection, (ii) free convection and (iii) combined free and forced convection.

(i) Tyagi [1–4] in a series of papers has studied the effect of viscous dissipation in forced convection through noncircular channels. He has used the method of complex variables and has obtained solutions for both Neumann and Dirichlet type thermal boundary conditions, showing that viscous dissipation has significant effect on the Nusselt number. Cheng [5] has studied the effect of viscous dissipation for flow through regular polygonal ducts using the method of point-matching.

(ii) Ostrach [6–9] has investigated the effects of viscous dissipation in natural convection flows through channels formed by two parallel long plane surfaces and has shown that the flow and heat transfer are not only functions of Prandtl and Grashof numbers but also depend on the dimensionless frictional heating parameter which may appreciably affect the mode of heat transfer.

(iii) The only published work in the field of combined free and forced convection is that of Ostrach [10, 11]. He has used the method of successive approximations to analyse the problem of taking into account the effects of frictional heating in flow between vertical parallel plane surfaces and has obtained results similar to his free convection analysis. The effects of viscous dissipation for flow through circular ducts has been recently reported [12] and shows that the effect of viscous dissipation on Nusselt numbers is not significant for small values of the dissipation parameter. The present study treats the case of annular flow and shows that the effect of viscous dissipation on Nusselt numbers may not be ignored, for the same value of the dissipation parameter.

### FORMULATION OF THE PROBLEM AND SOLUTION

Consider a vertical straight circular concentric annulus of inner and outer radii $r_i$ and $r_0$. The flow is considered to be laminar and fully developed, both hydrodynamically and thermally, and is in the vertical upward direction along the positive $z$-axis. The condition of uniform heat input per unit length in the direction of flow is considered. The fluid properties will be considered constant except for the variation of density in buoyancy term of the equation of motion. Under the above assumptions, all the terms in the differential form of the continuity equation will reduce identically equal to zero. Due to symmetry, the momentum equation in angular direction will not exist. The momentum equation in radial direction can be considered negligible. The equation of motion in $z$-direction can be written as

$$0 = -\frac{\mathrm{d}p}{\mathrm{d}z} + \mu\left(\frac{\mathrm{d}^2 v_2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}v_2}{\mathrm{d}r}\right) - \rho g. \quad (1)$$

The simplified energy equation can be written as,

$$\rho C_p v_z \frac{\partial T}{\partial z} = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \mu\left(\frac{\mathrm{d}v_2}{\mathrm{d}r}\right)^2 + T\beta v_2 \frac{\mathrm{d}P}{\mathrm{d}Z}. \quad (2)$$

The last two terms on the right-hand side of (2) are the viscous dissipation and compression work terms respectively. The compression work term as written above is obtained after simplifying the energy equation as given by Rohsenow and Choi [13]. Tyagi [3, 4] has shown the relative significance of these two terms. In the present study we will ignore the compression work term, this being smaller than the viscous dissipation term. For the condition of uniform heat input per unit length and constant fluid properties, the axial temperature gradient at the wall and for the fluid become constant and equal. Thus $\partial T/\partial Z = C$, where $C$ is a constant.

In non-dimensional form, equations (1) and (2) can be written as,

$$\left(\frac{\mathrm{d}^2 V}{\mathrm{d}R^2} + \frac{1}{R}\frac{\mathrm{d}V}{\mathrm{d}R}\right) + Ra\phi + L = 0, \quad (3)$$

$$\left(\frac{\mathrm{d}^2 \phi}{\mathrm{d}R^2} + \frac{1}{R}\frac{\mathrm{d}\phi}{\mathrm{d}R}\right) - V + 4M\left(\frac{\mathrm{d}V}{\mathrm{d}R}\right)^2 = 0. \quad (4)$$

In equations (3) and (4). Rayleigh number $Ra$ and the viscous dissipation parameter $M$ are prescribed quantities, while $V$, $\phi$ and $L$are the three unknowns to be determined. As there are two equations in three unknowns, we need another relation which is provided by the integral form of the continuity equation,

$$\iint V\mathrm{d}A = \iint \mathrm{d}A. \quad (5)$$

We have not so far mentioned anything about the boundary conditions. The condition of no slip at both walls will apply, giving

$$V = 0 \quad \text{at} \quad R = R_i \quad \text{and} \quad R = R_0. \quad (6)$$

The heat input can be supplied through either one or both the walls. We will consider the following three situations.

Case 1. Outer wall heated and inner adiabatic,

$$\phi = 0 \quad \text{at} \quad R = R_0; \quad \text{and} \quad \mathrm{d}\phi/\mathrm{d}R = 0 \quad \text{at} \quad R = R_i \quad (7)$$

Case 2. Inner wall heated and outer adiabatic,

$$\phi = 0 \quad \text{at} \quad R = R_i, \quad \text{and} \quad \mathrm{d}\phi/\mathrm{d}R = 0 \quad \text{at} \quad R = R_0. \quad (8)$$

Case 3. Both walls heated with equal temperatures,

$$\phi = 0 \quad \text{at} \quad R = R_i \quad \text{and} \quad R = R_0. \quad (9)$$

An exact solution of (3)–(6) with any one of the three thermal boundary conditions is extremely difficult, if at all possible. Main difficulty lies due to the nonlinear term in (4). This system of equations have been solved by the numerical integration method of Runge–Kutta of order four. This numerical integration method, however, utilizes starting initial values from an exact solution of the equations without viscous dissipation term. We will therefore, first present this exact solution with $M = 0$.

A general form of the exact solution without viscous dissipation [14] for the concentric annulus in the form of Kelvin functions is presented in a simplified manner for Case 1 only, the approach for the other two cases being similar. Neglecting viscous dissipation, ($M = 0$), we divide equations (3) and (4) by $L$; so that these equations will now contain only two unknown functions $\bar{V}$and $\bar{\phi}$, where $\bar{V} = V/L$ and $\bar{\phi} = \phi/L$ The solution of the resulting equation for circular concentric annulus can be written as

$$\bar{V} = C_1 ber_0(\eta R) + C_2 bei_0(\eta R) + C_3 ker_0(\eta R) + C_4 kei_0(\eta R), \quad (10)$$

$$\bar{\phi} = -\frac{1}{\eta^4} + \frac{1}{\eta^2}[-C_1 bei_0(\eta R) + C_2 ber_0(\eta R) - C_3 kei_0(\eta R) + C_4 ker_0(\eta R)]. \quad (11)$$

The unknowns, $C_1$, $C_2$, $C_3$ and $C_4$ are obtained by applying the boundary conditions, (6) and (7). This results in the following four equations,

$$0 = C_1 ber_0(\eta R_i) + C_2 bei_0(\eta R_i) + C_3 ker_0(\eta R_i) + C_4 kei_0(\eta R_i), \quad (12)$$

$$0 = -C_1 bei_0(\eta R_i) + C_2 ber_0(\eta R_i) - C_3 kei_0(\eta R_i) + C_4 ker_0(\eta R_i), \quad (13)$$

$$0 = C_1 ber_0(\eta R_0) + C_2 bei_0(\eta R_0) + C_3 ker_0(\eta R_0) + C_4 kei_0(\eta R_0), \quad (14)$$

$$1 = \eta^2[-C_1 bei_0(\eta R_0) + C_2 ber_0(\eta R_0) - C_3 kei_0(\eta R_0) + C_4 ker_0(\eta R_0). \quad (15)$$

Equations (12)–(15) are solved simultaneously to determine the values of the unknown constants, $C_1$, $C_2$, $C_3$ and $C_4$. Thus $\bar{V}$and $\phi$ can be evaluated and the pressure drop parameter $L$ is obtained from the continuity equation (5) as, $L = \iint \mathrm{d}A/\iint \bar{V}\mathrm{d}A$. Once $L$ is obtained, $V$ and $\phi$ are determined.

The numerical integration method of Runge–Kutta of order four was used to obtain the solutions with viscous dissipation term. The method requires the complete set of functional values $V$, $\phi$ and their gradients at the starting boundary point and the estimates of the missing initial boundary conditions were made from the exact solution results. The resulting solutions were then improved by iteration to obtain the desired solutions.

Once the solution for the velocity and temperature functions is obtained, Nusselt numbers can be evaluated. The Nusselt number expression including the viscous dissipation effect can be written in the dimensionless form as,

$$Nu = \frac{8M \int_{R_i}^{R_0} (\mathrm{d}V/\mathrm{d}R)^2 R \,\mathrm{d}R - 1}{\int_{R_i}^{R_0} V\phi R \,\mathrm{d}R / \int_{R_i}^{R_0} VR \,\mathrm{d}R}. \quad (16)$$

## DISCUSSIONS

The exact solution with $M = 0$ for the concentric annulus involved the Kelvin functions and their derivatives because of the thermal boundary condition of one wall being insulated. These functions were evaluated in Double Precision from McLachlan [15]. The number of terms required for convergence was of the order of 20. The nonlinear problem ($M > 0$) was solved by Runge–Kutta fourth order method in Double Precision and the accuracy of the R–K method was judged by obtaining results for $M = 0$ and comparing them with the exact solution results. These results agreed very closely.

Since the effect of viscous dissipation on the velocity and temperature field is found to be very small (for maximum value of $M = 0.0005$ used here), it is not convenient to present the results graphically and, therefore, only a few general observations will be made.

For pure convection ($Ra = 0$), the equation of motion (3) becomes independent of the energy equation (4). As such, the velocity field is unaffected by the presence of viscous dissipation term. The temperature field, however, is strongly influenced by the viscous dissipation parameter. This parameter $M$ converts viscous work to heat. When heat is being supplied to the duct externally, the parameter $M$ reduces the temperature differences $\phi$ in the fluid and this has been noted in all the three Cases treated here.

For $Ra > 0$, the equations (3) and (4) become coupled. This coupling becomes stronger as the value of $Ra$ is increased. Therefore, in combined free and forced convection, the viscous dissipation could strongly influence the velocity and temperature fields. In the present study of heated vertical ducts, viscous dissipation reduces the temperature differences. The reduced temperature differences influence the velocity profiles in a direction opposite to the free convection effect. In upflow heating, the effect of free convection is to increase the velocity near heated wall and reduce it elsewhere; while the effect of $M$ has been found to be precisely the opposite in all the three Cases studied here.

As mentioned earlier, viscous dissipation opposes the impressed external heating and reduces the heat transfer rate resulting in lower values of Nusselt numbers. Figure 1 shows the effect of viscous dissipation on Nusselt numbers

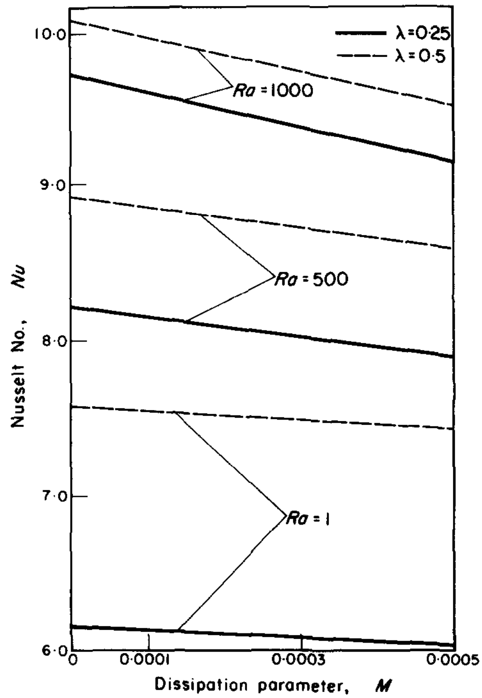


FIG. 1. Effect of viscous dissipation parameter on Nusselt number for concentric annulus with outer wall heated, inner wall insulated.

for outer wall heated, inner wall insulated with $\lambda = 0.25$ and 0·5. From this figure it can be seen that Nusselt numbers decrease with increase in the dissipation parameter $M$. The reduction in Nusselt numbers with increasing values of M becomes more pronounced at higher Rayleigh numbers. Figure 2 shows the effect of $M$ on Nusselt numbers for inner wall heated, outer wall insulated. For this case too, it can be seen that lower values of Nusselt numbers are obtained when viscous dissipation is taken into account. The effect of $M$ on Nusselt numbers for the case of both walls heated is shown in Fig. 3. As anticipated the Nusselt numbers are again reduced with increasing $M$ and this reduction becomes more pronounced at higher Rayleigh numbers.

The effect of radius ratio $\lambda$ on the Nusselt numbers can be seen from Figs. 1–3. Figures 1 and 3 show that for outer wall heated and inner wall insulated or for both walls heated,

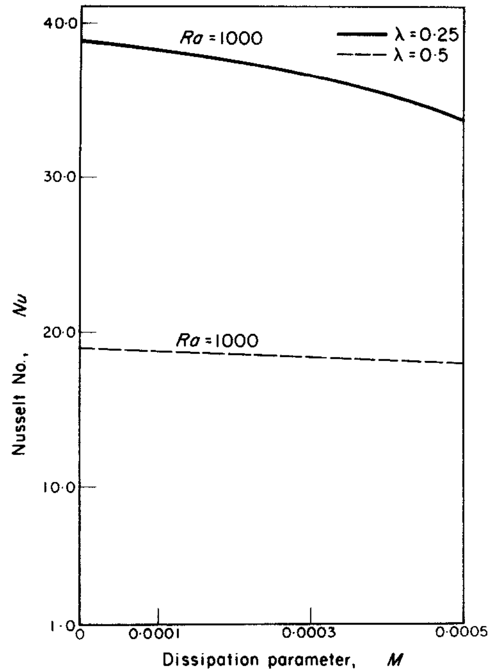


FIG. 2. Effect of viscous dissipation parameter on Nusselt number for concentric annulus with inner wall heated, outer wall insulated.

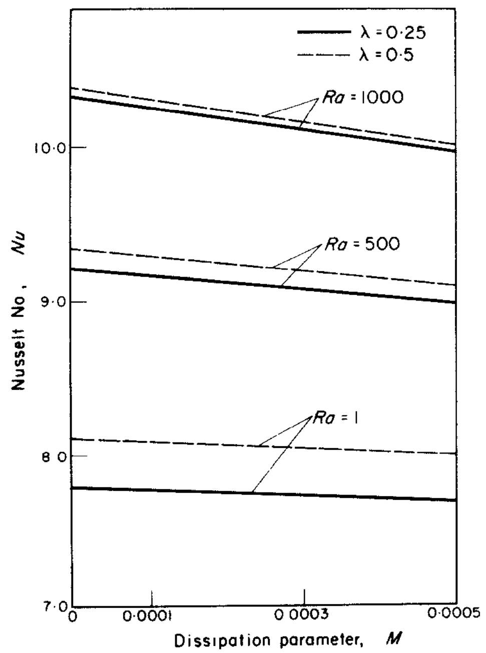


FIG. 3. Effect of viscous dissipation parameter on Nusselt number for concentric annulus with both walls heated.

high values of Nusselt numbers are obtained by increasing $\lambda$. Whereas from Fig. 2 it can be seen that, for inner wall heated, outer wall insulated, the Nusselt numbers are reduced by increasing $\lambda$. The reason for the latter case is due to the fact that by increasing $\lambda$, we in fact reduce the hydraulic diameter $D_h$, which in turn reduces the Nusselt numbers.

A comparison of the reduction in Nusselt numbers for the same value of the dissipation parameter $M$ has also been studied. It is found that the maximum reduction occurs for the case of inner wall heated, outer wall insulated and the minimum reduction occurs for the case of both walls heated.

## ACKNOWLEDGEMENTS

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# HEAT TRANSFER IN THE LAMINAR CREEPING FLOW BETWEEN PARALLEL CIRCULAR DISKS WITH ECCENTRIC INLET

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## NOMENCLATURE

| | |
|---|---|
| $b$, | space between disks; |
| $b_{ni}$, | coefficients in the expansions of the eigenfunctions; |
| $c$, | defined in equation (3); |
| $\bar{c}$, | $c/(b/2)$; |
| $c_i$, | defined in equation (18); |
| $C_p$, | specific heat at constant pressure; |
| $e$, | eccentricity; |
| $\bar{e}$, | $e/r_i$; |
| $E$, | Eckert number, $U_0^2/C_p\Delta T$; |
| $h, h_1, h_2$, | the scale factors defined in equation (2); |
| $k$, | heat conductivity; |
| $k_i, k_e$, | values of $x_2$ at inlet and outlet radii; |
| $N$, | a positive integer; |
| $Nu, \overline{Nu}$, | local and average Nusselt numbers; |
| $Pe$, | Péclét number; |
| $\overline{Pe}$, | $Pe\log_e(1/\gamma)/(k_i - k_e)$; |
| $q$, | heat transfer; |
| $r$, | radius of the disk; |
| $\overline{r_e}$, | $r_e/(b/2)$; |
| $\overline{r_i}$, | $r_i/(b/2)$; |
| $R$, | a function of $x_2$ only, equation (13); |
| $T$, | temperature; |
| $U_0$, | reference velocity; |
| $v_1, v_2$, | velocity components in the $x_1$- and $x_2$-directions; |
| $x_1, x_2, x_3$, | bipolar coordinate frame; |
| $x_2', \bar{x}_2, \bar{x}_3$; | defined in equations (10) and (11): |
| $\bar{Y}_i$, | defined in equation (19); |
| $Z$, | a function of $\bar{x}_3$ only, equation (13). |

Greek letters

| | |
|---|---|
| $\mu$, | dynamic viscosity; |
| $\rho$, | density; |
| $\gamma$, | $r_i/r_e$; |
| $\phi$, | $\bar{e}\gamma/(1-\gamma)$; |
| $\lambda_i, \lambda$, | eigenvalues; |
| $\theta$, | $(T_w - T)/(T_w - T_0)$. |

Subscripts $i$, $e$ and $w$ refer to conditions at inlet, exit and wall respectively.

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## ANALYSIS

THE ENRGY equation for the laminar, creeping flow of incompressible fluids between parallel circular disks with eccentric inlet (see Fig. 1) can be written, in bipolar coordinates, as

$$\rho C_p\left(\frac{v_1}{h_1}\frac{\partial T}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial T}{\partial x_2}\right) = k\frac{\partial^2 T}{\partial x_3^2} \tag{1}$$

subject to the boundary conditions:

$$T(x_1, x_2, b/2) = T(x_1, x_2, -b/2) = T_w$$
$$T(x_1, k_i, x_3) = T_0.$$

The viscous dissipation terms are neglected, which is justified for small Eckert numbers, i.e. $E(= U_m^2/C_p\Delta T) \ll 1$.